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A formula for gap between two closed operators

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ABSTRACT

Aim of this short note is to obtain a formula for the gap between two densely defined unbounded closed operators. It is interesting to note that the formula is very similar to the corresponding formula for the gap between two bounded operators.

1. Introduction

Let H denote an infinite dimensional complex Hilbert space. For a closed subspace M of H denote by P_M, the orthogonal projection onto M. The gap between closed subspaces M and N of H is given by $\delta(M,N):=\|P_M-P_N\|$. This defines a metric on the class of closed subspaces of H, known as the gap metric. The topology induced by the gap metric is known as the gap topology.

Next, let $H_1,\ H_2$ be two Hilbert spaces. If $A,B:H_1\rightarrow H_2$ are bounded operators, then the gap $\theta(A,B)$ between A and B is defined as the gap between the corresponding graphs $G(A)$ and $G(B)$. That is $\theta(A,B):=\|P_{G(A)}-P_{G(B)}\|$. In particular, if $B=0$, then $\theta(A):=\theta(A,0)$ is called the gap of the operator A.

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For an $m \times n$ matrix A, Habibi [4] proved that $\theta(A) = \frac{\|A\|}{\sqrt{1+\|A\|^2}}$. A formula for computing the gap between bounded operators $A, B \in \mathcal{B}(H)$ was deduced by Nakamoto [11], which generalizes Habibi's result. He proved the following:

$$\theta(A,B) = \max\left\{\left\|(I+BB^*)^{-1/2}(A-B)(I+A^*A)^{-1/2}\right\|, \left\|(I+AA^*)^{-1/2}(A-B)(I+B^*B)^{-1/2}\right\|\right\},$$

and also has shown that the topology induced by the gap metric and the topology induced by the operator norm on $\mathcal{B}(H)$ are the same.

Similarly if A, B are densely defined closed operators, then $G(A)$ and $G(B)$ are closed subspaces of $H_1 \times H_2$. Hence in this case also the gap between A and B is defined as $\|P_{G(A)} - P_{G(B)}\|$. This defines a metric on the class of closed operators and induces a topology also known as the gap topology. It is well known that the gap topology restricted to the class of bounded operators, coincides with the norm topology. Also the convergence with respect to the gap metric on the set of self-adjoint bounded operators coincide with the resolvent convergence [14, Chapter VII, p. 235].

In the literature the class of unbounded closed operators has not received due attention, although many operators that arise in physical applications are unbounded. The main difficulty in dealing with the unbounded operators is that they are usually defined on some proper subspace of a Hilbert space. Hence many techniques of bounded operators do not work for unbounded operators.

The aim of this article is to generalize the results of Habibi [4] (see Theorem 3.2) and Nakamoto [11] to the case of unbounded closed operators (see Theorem 3.5).

The next section contains some preliminary results. In the third section we obtain a formula for the gap of a closed operator and generalize a theorem of Nakamoto [11, Theorem 1.1] to the class of unbounded operators.

It is a remarkable fact to observe that this formula is essentially same as the formula for the gap between two bounded operators. This is due to the fact that the operators that appear in the formula (for the gap between unbounded closed operators) are bounded.

In [9], the author has defined a metric on a class of operators on a Hilbert C^*-module. This metric is equivalent to the gap metric, but it is not the same (See Remark 3.6 for details).

2. Notations and basic results

Throughout the paper we consider complex Hilbert spaces which will be denoted by H, H_1, H_2, etc. The inner product and the induced norm are denoted by $\langle , \rangle$ and $||.||$, respectively. If $T : H_1 \to H_2$ is a linear operator with domain $D(T) \subseteq H_1$, then it is denoted by $T \in \mathcal{L}(H_1, H_2)$. The null space and the range space of T are denoted by $N(T)$ and $R(T)$, respectively.

The graph of $T \in \mathcal{L}(H_1, H_2)$ is defined by $G(T) := \{(x, Tx) : x \in D(T)\} \subseteq H_1 \times H_2$. If $G(T)$ is closed, then T is called a **closed operator**. The set of all closed operators between H_1 and H_2 is denoted by $\mathcal{C}(H_1, H_2)$. By the closed graph Theorem [5, p. 281], an everywhere defined closed operator is bounded. The set of all bounded operators is denoted by $\mathcal{B}(H_1, H_2)$. If $H_1 = H_2 = H$, then $\mathcal{B}(H_1, H_2)$ and $\mathcal{C}(H_1, H_2)$ are denoted by $\mathcal{B}(H)$ and $\mathcal{C}(H)$, respectively.

If S and T are closed operators with the property that $D(S) \subseteq D(T)$ and $Sx = Tx$ for all $x \in D(S)$, then S is called the restriction of T and T is called the extension of S. If M is a closed subspace of a Hilbert space H, then $M^{\perp}$ is the orthogonal complement of M in H.

For closed subspaces M_1 and M_2 of H, the direct sum and the orthogonal direct sum are denoted by $M_1 \oplus M_2$ and $M_1 \oplus^{\perp} M_2$, respectively. Throughout we denote that if $T \in \mathcal{C}(H_1, H_2)$ is densely defined, then $\check{T} = (I + T^*T)^{-1}$ and $\widehat{T} = (I + TT^*)^{-1}$.

Here we recall some of the known facts which will be used in the next section.

Proposition 2.1 [1, p. 70]. *Let M, N be closed subspaces of a Hilbert space H. Then*

$$\delta(M,N) := ||P_M - P_N|| = \max\{||P_M(I - P_N)||, ||P_N(I - P_M)||\}.$$

Proposition 2.2 [3, pp. 70–71]. *Let H_i be Hilbert spaces and $A_i \in \mathcal{B}(H_i)$, $i = 1, 2$. Let $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$. Then $A \in \mathcal{B}(H_1 \oplus H_2)$ and $||A|| = \max\{||A_1||, ||A_2||\}$.*

Definition 2.3 *[15, Theorem 13.31, p. 349].* Let $T \in \mathcal{C}(H)$ be a positive operator. Then there exists a unique positive operator S such that $T = S^2$. The operator S is called the square root of T and is denoted by $S = T^{\frac{1}{2}}$.

Proposition 2.4 [15, Theorem 13.13, p. 336]. *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then the operator $I + T^*T : D(T^*T) \to H_1$ is bijective and $\check{T}$ is bounded.*

Lemma 2.5 [6,7,13]. *Let $A \in \mathcal{C}(H_1, H_2)$ be densely defined. Then*

1. $\check{A} \in \mathcal{B}(H_1), \widehat{A} \in \mathcal{B}(H_2)$.
2. $\widehat{A}A \subseteq A\check{A}$, $\quad ||A\check{A}|| \leqslant \frac{1}{2}$ *and* $\check{A}A^* \subseteq A^*\widehat{A}$, $\quad ||A^*\widehat{A}|| \leqslant \frac{1}{2}$.

Lemma 2.6 [2, Lemma 5.1]. *Let $A \in \mathcal{C}(H_1, H_2)$ be densely defined. Then*

1. $\check{A}^{\frac{1}{2}}$ *and* $A\check{A}^{\frac{1}{2}}$ *are bounded.*
2. $\left\|\check{A}^{\frac{1}{2}}\right\| \leqslant 1$ *and* $\left\|A\check{A}^{\frac{1}{2}}\right\| \leqslant 1$.

3. The gap between closed operators

Recall that if $A, B \in \mathcal{C}(H_1, H_2)$, then the gap between A and B is defined by $\theta(A, B) = \|P_{G(A)} - P_{G(B)}\|$, where $P_M : H_1 \times H_2 \to H_1 \times H_2$ is an orthogonal projection onto the closed subspace M of $H_1 \times H_2$. In this section first we obtain a formula for the gap of a closed operator which generalizes the result of [4]. For this purpose we use the matrix representation of the orthogonal projection onto the graph of a closed operator. This representation was obtained in [2]. Also see [12] and [16, pp. 72–73]. This result for self-adjoint bounded operators was proved by Halmos using elementary geometric concepts in [8].

Theorem 3.1 [2,16]. *Let $A \in \mathcal{C}(H_1, H_2)$ be densely defined. Let $P := P_{G(A)}$. Then*

$$P = \begin{bmatrix} \check{A} & A^*\widehat{A} \\ A\check{A} & I - \widehat{A} \end{bmatrix}.$$

Similarly if $B \in \mathcal{C}(H_1, H_2)$ is densely defined and $Q = P_{G(B)}$, then

$$\theta(A, B) = \|P - Q\| = \left\| \begin{bmatrix} \check{A} - \check{B} & A^*\widehat{A} - B^*\widehat{B} \\ A\check{A} - B\check{B} & \widehat{B} - \widehat{A} \end{bmatrix} \right\|.$$

Theorem 3.2. *Let $A \in \mathcal{C}(H_1, H_2)$ be densely defined. Then $A\check{A}^{\frac{1}{2}}$ is bounded and*

$$\theta(A, 0) = ||A\check{A}^{\frac{1}{2}}||.$$

Proof. Note that $A\check{A}^{\frac{1}{2}}$ is bounded by Lemma 2.6. Let $P := P_{G(A)}$ and $Q := P_{G(0)}$. Then by Proposition 2.1,

$$\theta(A, 0) = \max\{||P(I - Q)||, \ ||Q(I - P)||\}.$$

Now,

$$I - Q = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix}.$$

$$P(I-Q)=\begin{bmatrix}\check{A} & A^*\widehat{A}\\ A\check{A} & AA^*\widehat{A}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & -I\end{bmatrix}=\begin{bmatrix}0 & -A^*\widehat{A}\\ 0 & -AA^*\widehat{A}\end{bmatrix}:=T \quad (\text{say}).$$

Then

$$T^*=\begin{bmatrix}0 & 0\\ -\widehat{A}A & -\widehat{A}AA^*\end{bmatrix} \text{ and } T^*T=\begin{bmatrix}0 & 0\\ 0 & \widehat{A}AA^*\widehat{A}+\widehat{A}AA^*AA^*\widehat{A}\end{bmatrix}.$$

Therefore, using the relations in Lemma 2.5 and Proposition 2.2,

$$\begin{aligned}||T||^2=||T^*T|| &= \max\{||0||,||\widehat{A}AA^*\widehat{A}+\widehat{A}AA^*AA^*\widehat{A}||\}\\ &=||\widehat{A}AA^*\widehat{A}+\widehat{A}AA^*AA^*\widehat{A}||\\ &=||\widehat{A}AA^*(I+AA^*)\widehat{A}||\\ &=||\widehat{A}AA^*||.\end{aligned}$$

Hence $||T||=||\widehat{A}AA^*||^{\frac{1}{2}}=||A(I+A^*A)^{\frac{-1}{2}}||=\left\|A\check{A}^{\frac{1}{2}}\right\|$.

A similar computations ensures that $||Q(I-P)||=||A\check{A}^{\frac{1}{2}}||$. This completes the proof. □

Remark 3.3. Let $A\in\mathcal{B}(H_1,H_2)$. Then it is easy to show that $\left\|A\check{A}^{\frac{1}{2}}\right\|=\frac{\|A\|}{\sqrt{1+\|A\|^2}}$. Hence by Theorem 3.2, the gap of A is given by

$$\theta(A):=\theta(A,0)=\frac{||A||}{\sqrt{1+||A||^2}}.$$

In particular, when A is an $m\times n$ matrix we obtain the result of Habibi [4].

Next we obtain a formula for $\theta(A,B)$ where both $A\neq 0$ and $B\neq 0$. Let $A,B\in\mathcal{C}(H_1,H_2)$ be densely defined. Recall that $\widehat{T}=(I+TT^*)^{-1}$ and $\check{T}=(I+T^*T)^{-1}$ for any densely defined $T\in\mathcal{C}(H_1,H_2)$.

Lemma 3.4 [1, p. 70]. *Let $P:=P_{G(A)}$ and $Q:=P_{G(B)}$. Then*

1. $\|P-Q\|=\max\{\|P(I-Q)\|,\|Q(I-P\|)\}$.
2. $\|PQ\|=\sup\left\{\frac{|\langle x,y\rangle|}{\|x\|\ \|y\|}:Px=x\neq 0,\ Qy=y\neq 0\right\}$.

Theorem 3.5. *Let $A,B\in\mathcal{C}(H_1,H_2)$ be densely defined. Then the operators $\widehat{B}^{\frac{1}{2}}A\check{A}^{\frac{1}{2}}$, $B\check{B}^{\frac{1}{2}}\check{A}^{\frac{1}{2}}$, $A\check{A}^{\frac{1}{2}}\check{B}^{\frac{1}{2}}$ and $\widehat{A}^{\frac{1}{2}}B\check{B}^{\frac{1}{2}}$ are bounded and*

$$\theta(A,B)=\max\left\{\left\|B\check{B}^{\frac{1}{2}}\check{A}^{\frac{1}{2}}-\widehat{B}^{\frac{1}{2}}A\check{A}^{\frac{1}{2}}\right\|,\left\|A\check{A}^{\frac{1}{2}}\check{B}^{\frac{1}{2}}-\widehat{A}^{\frac{1}{2}}B\check{B}^{\frac{1}{2}}\right\|\right\}.$$

Proof. Let $P,\ Q$ be as in Lemma 3.4. First note that $\widehat{B}^{\frac{1}{2}}A\check{A}^{\frac{1}{2}}$ is bounded since it is the composition of two bounded operators $\widehat{B}^{\frac{1}{2}}$ and $A\check{A}^{\frac{1}{2}}$. A similar argument concludes the boundedness of the operators $B\check{B}^{\frac{1}{2}}\check{A}^{\frac{1}{2}}$, $A\check{A}^{\frac{1}{2}}\check{B}^{\frac{1}{2}}$ and $\widehat{A}^{\frac{1}{2}}B\check{B}^{\frac{1}{2}}$.

Observe that $I-Q$ is an orthogonal projection onto the subspace $\{(-B^*y,y):y\in D(B^*)\}$. Now let us calculate $\|P(I-Q)\|$ using Lemma 3.4. By definition,

$$\|P(I-Q)\|=\sup_{0\neq x\in D(A),\,0\neq y\in D(B^*)}\left\{\frac{|\langle(x,Ax),(-B^*y,y)\rangle|}{\|(x,Ax)\|\ \|(-B^*y,y)\|}\right\}$$

$$= \sup_{o\neq x\in D(A),\, 0\neq y\in D(B^*)} \left\{ \frac{|\langle x, -B^*y\rangle + \langle Ax, y\rangle|}{\sqrt{\|x\|^2 + \|Ax\|^2}\sqrt{\|y\|^2 + \|B^*y\|^2}} \right\}.$$

The maps $\check{A}^{\frac{1}{2}} : H_1 \to D(A)$ and $\widehat{B}^{\frac{1}{2}} : H_2 \to D(B^*)$ are bijective. Hence for every $0 \neq x \in D(A)$ there exists a unique $0 \neq u \in H_1$ such that $x = \check{A}^{\frac{1}{2}}u$. For a similar reason for every $0 \neq y \in D(B^*)$ there exists a unique $0 \neq v \in H_2$ such that $y = \widehat{B}^{\frac{1}{2}}v$.

Now

$$\begin{aligned}
\|x\|^2 + \|Ax\|^2 &= \langle x, x\rangle + \langle Ax, Ax\rangle \\
&= \langle \check{A}^{\frac{1}{2}}u, \check{A}^{\frac{1}{2}}u\rangle + \langle A\check{A}^{\frac{1}{2}}u, A\check{A}^{\frac{1}{2}}u\rangle \\
&= \langle \check{A}u, u\rangle + \langle A^*A\check{A}u, u\rangle \\
&= \langle u, u\rangle \\
&= \|u\|^2.
\end{aligned}$$

Similarly $\|y\|^2 + \|B^*y\|^2 = \|v\|^2$. Hence

$$\begin{aligned}
\|P(I-Q)\| &= \sup_{0\neq u\in H_1,\, 0\neq v\in H_2} \left\{ \frac{\left|\left\langle \check{A}^{\frac{1}{2}}u, -B^*\widehat{B}^{\frac{1}{2}}v\right\rangle + \left\langle A\check{A}^{\frac{1}{2}}u, \widehat{B}^{\frac{1}{2}}v\right\rangle\right|}{\|u\|\,\|v\|} \right\} \\
&= \sup_{0\neq u\in H_1,\, 0\neq v\in H_2} \left\{ \frac{\left|\left\langle B\check{B}^{\frac{1}{2}}\check{A}^{\frac{1}{2}}u, v\right\rangle - \left\langle \widehat{B}^{\frac{1}{2}}A\check{A}^{\frac{1}{2}}u, v\right\rangle\right|}{\|u\|\,\|v\|} \right\} \\
&= \sup_{0\neq u\in H_1,\, 0\neq v\in H_2} \left\{ \frac{\left|\left\langle \left(B\check{B}^{\frac{1}{2}}\check{A}^{\frac{1}{2}} - \widehat{B}^{\frac{1}{2}}A\check{A}^{\frac{1}{2}}\right)u, v\right\rangle\right|}{\|u\|\,\|v\|} \right\} \\
&= \left\| B\check{B}^{\frac{1}{2}}\check{A}^{\frac{1}{2}} - \widehat{B}^{\frac{1}{2}}A\check{A}^{\frac{1}{2}} \right\|.
\end{aligned}$$

By a similar argument it follows that

$$\|Q(I-P)\| = \left\| A\check{A}^{\frac{1}{2}}\check{B}^{\frac{1}{2}} - \widehat{A}^{\frac{1}{2}}B\check{B}^{\frac{1}{2}} \right\|.$$

Thus the gap between A and B is given by
$\theta(A, B) = \max\left\{ \left\| B\check{B}^{\frac{1}{2}}\check{A}^{\frac{1}{2}} - \widehat{B}^{\frac{1}{2}}A\check{A}^{\frac{1}{2}} \right\|, \left\| A\check{A}^{\frac{1}{2}}\check{B}^{\frac{1}{2}} - \widehat{A}^{\frac{1}{2}}B\check{B}^{\frac{1}{2}} \right\| \right\}.$ □

Remark 3.6. In [9, Eqs. (2.2) and (2.3)], the author has defined a metric d on the class of unbounded regular operators on Hilbert C^*-modules (see [10, Chapter 9] for the details of such operators). This metric is equivalent to the gap metric, but it is different. This formula for the case of densely defined unbounded closed operators in Hilbert spaces reduces to the following:

Let $A, B \in \mathcal{C}(H_1, H_2)$ be densely defined. Then

$$d(A, B) := \sup\{\|\check{A} - \check{B}\|,\ \|\widehat{A} - \widehat{B}\|,\ \|A\check{A} - B\check{B}\|\}. \tag{3.1}$$

In particular, $d(A, 0) = \|A\check{A}\| \neq \|A\check{A}^{\frac{1}{2}}\| = \theta(A)$. Thus the formula 3.1 and the gap formula *are not the same*.

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